Asymptotic M5-brane entropy from S-duality

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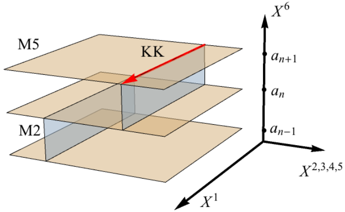
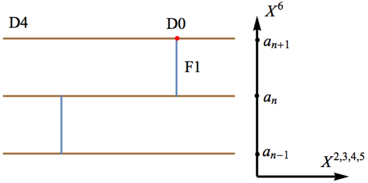
- Tallk based on: S.Kim and J.Nahmgoong [work in progress]

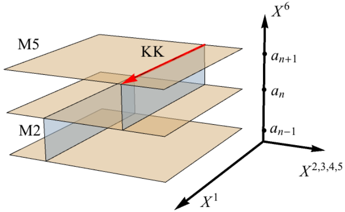
6d (0,2) SCFT

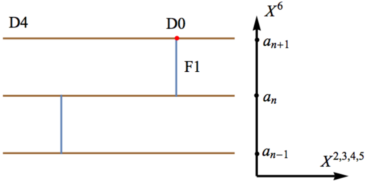




- Worldvolume theory of coincident N M5-branes $\rightarrow$ 6d (0,2) A_{N-1} SCFT
  - Entropy $\propto N^3$
  - No known Lagrangian description
- Reducing 6d over S^1: 5d $\mathcal{N}=2$ SYM + instanton (KK mode)
  - M-theory circle $X_1 \sim X_1 + 2\pi R_1$
  - Hypermultiplet mass m: $\mathcal{N}=2 \rightarrow \mathcal{N}=1^*$
- Separating M5 branes: Coulomb phase $U(N) \rightarrow U(1)^N$
  - Giving VEV to 5d vector multiplet $(A_\mu, \phi^{6,7,8,9,10})$
  - $\langle \phi^6 \rangle = \text{diag}(a_1, ..., a_N)$, $\langle \phi^{7,\dots,10} \rangle = 0$

5d $\mathcal{N} = 1^*$ SYM





- BPS objects of 5d SYM in Coulomb phase
  - W-boson: F1 connected between two D4s
  - Instanton: D0 bounded on a D4
- They form 1/4 BPS states counted by partition function
- Reducing 5d over temporal S^1: Ω-background
  - Temporal circle $X_0 \sim X_0 + 2\pi R_0$
  - Twisting parameter $\epsilon_{1,2}$: $z_{1,2} \sim e^{\epsilon_{1,2}} z_{1,2}, \quad w_{1,2} \sim e^{-\frac{\epsilon_1+\epsilon_2}{2}} w_{1,2}$

$$z_{1,2} \in \mathbb{C}^2_{\parallel} = \mathbb{R}^4_{\parallel}(X^{2,3,4,5}), \quad w_{1,2} \in \mathbb{C}^2_{\perp} = \mathbb{R}^4_{\perp}(X^{7,8,9,10})$$

  - Effective volume of the system $V \sim \frac{1}{\epsilon_1 \epsilon_2}$

5d $\mathcal{N} = 1^*$ partition function: I

- Partition function $Z = Z_{\text{pert}} \cdot Z_{\text{inst}}$
- Instanton partition function

$$Z_{\text{inst}}(\tau, a, m, \epsilon_{1,2}) = 1 + \sum_{k=1}^{\infty} Z_k(a, m, \epsilon_{1,2}) q^k, \quad q = e^{2\pi i \tau}$$

$$Z_k = \text{Tr}\Big[(-1)^F q^k e^{-\beta\{Q,Q^\dagger\}} e^{-2\epsilon_+(J_{1R}+J_{2R})} e^{-2\epsilon_- J_{1L}} e^{-2m J_{2L}} \prod_{i=1}^{N} e^{-2a_i \Pi_i}\Big]$$

  - $m, \epsilon_{1,2}$: Chemical potentials on T^2 ($\mathbb{R}^{5,1} \to \mathbb{R}^4 \times T^2$)
  - a_i: Coulomb VEV
  - τ: Complex structure of T^2, complex gauge coupling in 5d

$$\tau = i\frac{R_0}{R_1} = i\frac{4\pi R_0}{g_{YM}^2}$$

  - q: instanton number fugacity
  - Z_k: Witten index of 5d SYM on $\mathbb{R}^{4,1}$

5d $\mathcal{N} = 1^*$ partition function: II

- Instaton partition function
  - Z_k: computed from D0-D4 ADHM QM

$$Z_k = \frac{1}{k!}\oint_{\text{JK}}\left[\prod_{I=1}^{k}\frac{d\phi_I}{2\pi i}\right] Z_{\text{vec}}(\phi, a, \epsilon_{1,2}) \cdot Z_{\text{adj}}(\phi, a, m, \epsilon_{1,2})$$

  - $\phi_I = -2\pi i\tau(A_I^1 + iA_I^t)$: Complexified gauge holonomy
  - Contour: Jefferey-Kirwan residue
  - $Z_{\text{vec/adj}}$: Gaussian path integral over massive modes

$$Z_{\text{vec}} = \prod_{I,J}\frac{2\sinh\frac{\phi_{IJ}}{2}\cdot 2\sinh\frac{\phi_{IJ}+2\epsilon_+}{2}}{2\sinh\frac{\phi_{IJ}+\epsilon_1}{2}\cdot 2\sinh\frac{\phi_{IJ}+\epsilon_2}{2}}\cdot\prod_{I,i}\frac{1}{2\sinh\frac{\phi_I - a_i \pm\epsilon_+}{2}}$$

$$Z_{\text{adj}} = \prod_{I,J}\frac{2\sinh\frac{\phi_{IJ}\pm m-\epsilon_-}{2}}{2\sinh\frac{\phi_{IJ}\pm m-\epsilon_+}{2}}\prod_{I,i}2\sinh\frac{\phi_I - a_i \pm m}{2}$$

- Perturbative partition function: Only W-boson contribution

$$Z_{\text{pert}}(a, m, \epsilon_{1,2}) = PE\left[\frac{1}{2}\frac{\sinh\frac{m+\epsilon_+}{2}\sinh\frac{m-\epsilon_+}{2}}{\sinh\frac{\epsilon_1}{2}\sinh\frac{\epsilon_2}{2}}\sum_{\alpha\in\text{root}} e^{\alpha\cdot a}\right]$$

Prepotential

- We consider the limit $\epsilon_{1,2} \to 0$

$$Z(\tau, a, m, \epsilon_{1,2}) = \exp\Big[- \frac{F(\tau, a, m)}{\epsilon_1 \epsilon_2} + \mathcal{O}(\epsilon_{1,2}^0) \Big]$$

- Prepotential F
  - Effective action of Coulomb branch
  - Classical part + quantum corrections (pertuabative / instanton)

$$F_{\text{tot}} = \pi i \tau a_i^2 + F, \quad F = F_{\text{pert}} + F_{\text{inst}}$$

- Ω-background parameter $\epsilon_{1,2}$ dependence is removed in prepotential.
  - Easier then partition function to treat

S-duality

- We compactified 6d (0,2) theory on $\mathbb{R}^4 \times T^2$
- S-duality: Interchanging radii of M-theory circle $(R_1) \leftrightarrow$ temporal circle (R_0)

$$\tau = i\frac{R_0}{R_1}, \quad \tau^D = i\frac{R_1}{R_0} \quad \rightarrow \quad \tau^D = -\frac{1}{\tau}$$

- S-dual transform of chemical potentials

$$m^D = \frac{m}{\tau}, \quad \epsilon_{1,2}^D = \frac{\epsilon_{1,2}}{\tau}$$

- S-duality of 4d $\mathcal{N} = 2^*$ prepotential: Legendre transform

$$\frac{1}{\epsilon_1^D \epsilon_2^D} F_{\text{tot}}^{\text{4d}}(\tau^D, a^D, m^D) = \frac{1}{\epsilon_1 \epsilon_2}\left(F_{\text{tot}}^{\text{4d}}(\tau, a, m) - a_i \partial_i F_{\text{tot}}^{\text{4d}}\right), \quad a_i^D = \frac{1}{2\pi i \tau}\partial_i F_{\text{tot}}^{\text{4d}}$$

$$F^{\text{4d}}(\tau^D, a^D, m^D) = \frac{1}{\tau^2}\left(F^{\text{4d}}(\tau, a, m) + \frac{1}{4\pi i \tau}(\partial_i F^{\text{4d}})^2\right), \qquad a_i^D = a_i + \frac{1}{2\pi i \tau}\partial_i F^{\text{4d}}$$

- S-duality of 5d prepotential?
  - Suspected to be Legendre transform.
  - Perturbative check is impossible: F^D cannot be expanded in q
  - Test using Modular anomaly equation

Modular anomaly equation

- τ dependence of F: q-series
  - Eisenstein series E_{2n} can be used as a basis for q-series

$$E_{2n}(\tau) = 1 + \frac{2}{\zeta(1-2n)} \sum_{k=1}^{\infty} k^{2n-1} \frac{q^k}{1-q^k}$$
$$= 1 + \frac{2}{\zeta(1-2n)} q + \frac{4^n+2}{\zeta(1-2n)} q^2 + \mathcal{O}(q^3)$$

  - $E_{2n>2}$'s are exact-modular. Only E_2 is quasi-modular

$$E_{2n>2}(\tau^D) = \tau^{2n} E_{2n}(\tau), \quad E_2(\tau^D) = \tau^2 \Big(E_2(\tau) + \frac{6}{\pi i \tau} \Big)$$

- S-duality is determined by E_2 dependence: Modular anomaly equation

$$\boxed{(F^{\text{4d}})^D = \frac{1}{\tau^2} \Big(F^{\text{4d}} + \frac{1}{4\pi i \tau} (\partial_i F^{\text{4d}})^2 \Big) \longleftrightarrow \frac{\partial F^{\text{4d}}}{\partial E_2} = -\frac{1}{24} (\partial_i F^{\text{4d}})^2}$$

- By checking E_2 dependence of 5d prepotential, we can reconstruct its S-duality transform

5d $\mathcal{N} = 1^*$ prepotential: I

- Partition function

$$Z = Z_{\text{pert}}\Big(1 + Z_1 q + Z_2 q^2 + \mathcal{O}(q^3)\Big)$$

- 5d $\mathcal{N} = 1^*$ prepotential is given in series of q

$$\begin{aligned}
F = {} & 1 \cdot \sum_{i,j}\Big(\mathrm{Li}_3(e^{a_{ij}}) - \frac{1}{2}\mathrm{Li}_3(e^{a_{ij}\pm m})\Big) \\
& + q \cdot 4\sinh^2\frac{m}{2}\sum_i T_i \quad \text{where } T_i = \prod_{j\neq i}\Big(1 - \frac{\sinh^2\frac{m}{2}}{\sinh^2\frac{a_{ij}}{2}}\Big) \\
& + q^2 \cdot \left(\sinh^6(\frac{m}{2})\sum_{i,j}\frac{8\sinh^4\frac{a_{ij}}{2} + 12\sinh^2\frac{a_{ij}}{2} - 4\sinh^2\frac{m}{2}}{\sinh^2\frac{a_{ij}}{2}\sinh^2\frac{a_{ij}\pm m}{2}} T_i T_j \right. \\
& \left. + \sinh^2(\frac{m}{2})\Big(6 + 2\sinh^2\frac{m}{2}\Big)\sum_i T_i^2 + 4\sinh^4(\frac{m}{2})\sum_i T_i T_i^{(2)}\right) + \mathcal{O}(q^3)
\end{aligned}$$

- E_2 dependence is not obviously seen.
- We expand F in series of m, and resum over q

- We expand F in series of m,

$$
\begin{aligned}
F = &- m^2\Big(\frac{1}{2}\sum_{i,j}\mathrm{Li}_1(e^{a_{ij}}) + N(q + \frac{3}{2}q^2 + \ldots)\Big)\\
&- m^4\,\frac{1 - 24q - 72q^2 + \ldots}{24}\Big(\sum_{i,j}\mathrm{Li}_{-1}(e^{a_{ij}}) - \frac{N}{12}\Big)\\
&- m^6\Bigg(\frac{1 - 48q + 432q^2 + \ldots}{2304}\\
&\times\Big[\sum_{i,j}\frac{1+\sinh^2\frac{a_{ij}}{2}}{\sinh^4\frac{a_{ij}}{2}} + \sum_{i,j,k}\Big(\frac{1}{2} + \frac{\cosh\frac{a_{ij}}{2}}{2\sinh\frac{a_{ij}}{2}}\Big)\Big(\frac{\cosh\frac{a_{ik}}{2}}{\sinh^3\frac{a_{ik}}{2}} + \frac{\cosh\frac{a_{kj}}{2}}{\sinh^3\frac{a_{kj}}{2}}\Big)\Big]\\
&+ \frac{1 + 240q + 2160q^2 + \ldots}{11520}\\
&\times\Big[\sum_{i,j}\frac{1-\sinh^2\frac{a_{ij}}{2}}{\sinh^4\frac{a_{ij}}{2}} - 5\sum_{i,j,k}\Big(\frac{1}{2} + \frac{\cosh\frac{a_{ij}}{2}}{2\sinh\frac{a_{ij}}{2}}\Big)\Big(\frac{\cosh\frac{a_{ik}}{2}}{\sinh^3\frac{a_{ik}}{2}} + \frac{\cosh\frac{a_{kj}}{2}}{\sinh^3\frac{a_{kj}}{2}}\Big)\Big]\Bigg)\\
&+ \mathcal{O}(m^8)
\end{aligned}
$$

5d $\mathcal{N} = 1^*$ prepotential: II

- We expand F in series of m, and resum over q

$$F = -m^2\Big(\frac{1}{2}\sum_{i,j}\mathrm{Li}_1(e^{a_{ij}}) - N\ln\phi(q)\Big) - m^4\frac{E_2}{24}\Big(\sum_{i,j}\mathrm{Li}_{-1}(e^{a_{ij}}) - \frac{N}{12}\Big)$$

$$-\,m^6\Bigg(\frac{E_2^2}{2304}\Big[\sum_{i,j}\frac{1+\sinh^2\frac{a_{ij}}{2}}{\sinh^4\frac{a_{ij}}{2}} + \sum_{i,j,k}\Big(\frac{1}{2}+\frac{\cosh\frac{a_{ij}}{2}}{2\sinh\frac{a_{ij}}{2}}\Big)\Big(\frac{\cosh\frac{a_{ik}}{2}}{\sinh^3\frac{a_{ik}}{2}}+\frac{\cosh\frac{a_{kj}}{2}}{\sinh^3\frac{a_{kj}}{2}}\Big)\Big]$$

$$+\,\frac{E_4}{11520}\Big[\sum_{i,j}\frac{1-\sinh^2\frac{a_{ij}}{2}}{\sinh^4\frac{a_{ij}}{2}} - 5\sum_{i,j,k}\Big(\frac{1}{2}+\frac{\cosh\frac{a_{ij}}{2}}{2\sinh\frac{a_{ij}}{2}}\Big)\Big(\frac{\cosh\frac{a_{ik}}{2}}{\sinh^3\frac{a_{ik}}{2}}+\frac{\cosh\frac{a_{kj}}{2}}{\sinh^3\frac{a_{kj}}{2}}\Big)\Big]\Bigg)$$

$+\,\mathcal{O}(m^8)$ checked up to q^3 for generic N, q^4 for $N=2$

- m^{2n+2} order: quasi-modular form with weight $2n$

$$m^4 : E_2, \quad m^6 : E_2^2, E_4, \quad m^8 : E_2^3, E_2E_4, E_6, \quad m^{10} : E_2^4, E_2^2E_4, E_2E_6, E_4^2$$

  - Only E_2, E_4, E_6 are independent
  - The number of combinations are finite $\rightarrow$ Resumming q-series into Eisenstein series can be done uniquely

S-duality of 5d $\mathcal{N} = 1^*$ prepotential

- m^2 order,

$$F = -\, m^2\Big(\frac{1}{2}\sum_\alpha \mathsf{Li}_1(e^{\alpha\cdot a}) - N\ln\phi(q)\Big) - m^4\frac{E_2}{24}\Big(\sum_\alpha \mathsf{Li}_{-1}(e^{\alpha\cdot a}) - \frac{N}{12}\Big) + \ldots$$

- Euler totient function $\phi(q)$

$$\phi(q) = \prod_{n=1}^{\infty}(1-q^n) = q^{-\frac{1}{24}}\eta(q) \quad \rightarrow \quad \ln\phi^D = \ln\phi + \ln\sqrt{-i\tau} - \frac{\pi i(\tau^D - \tau)}{12}$$

- E_2 dependence of F

$$\frac{\partial F}{\partial E_2} = -\frac{1}{24}(\partial_i F)^2 + m^4\frac{N^3}{288}$$ checked up to m^6 for generic N, m^{10} for $N=2$

- S-duality of the 5d $\mathcal{N} = 1^*$ prepotential F

$$F^D = \frac{1}{\tau^2}\Big(F + \frac{1}{4\pi i\tau}(\partial_i F)^2 + m^4\frac{N^3}{48\pi i\tau} + m^2 N\Big(\frac{\pi i(\tau-\tau^D)}{12} + \ln\sqrt{\frac{i}{\tau}}\Big)$$

- S-duality of the 5d prepotential F

$$F^D = \frac{1}{\tau^2}\Big(F + \frac{1}{4\pi i\tau}(\partial_i F)^2 + m^4\frac{N^3}{48\pi i\tau} + m^2 N\Big(\frac{\pi i(\tau-\tau^D)}{12} + \ln\sqrt{\frac{i}{\tau}}\Big)$$

- Strong coupling for $\tau^D \longleftrightarrow$ Weak coupling for τ

$$\tau^D = i\cdot 0^+, \quad q^D = e^{2\pi i\tau^D} = 1^- \quad \longleftrightarrow \quad \tau = i\cdot\infty, \quad q = 0^+$$

- Weak coupling prepotential $\simeq$ Perturbative prepotential

$$F \simeq F_{\text{pert}}$$

- Leading terms are Coulomb VEV independent

$$F^D \simeq -\frac{1}{\tau^D}\left(\frac{N^3 m_D^4}{48\pi i} + \frac{N^2 m_D^3}{12} + N m_D^2\frac{\pi i}{12}\right)$$
$$\underbrace{=}_{0<\text{Im}[m^D]<\frac{2\pi}{N}} \frac{1}{N\pi i\tau^D}\Big(\frac{1}{2}\text{Li}_4(e^{Nm^D}) + \frac{1}{2}\text{Li}_4(e^{-Nm^D}) - \text{Li}_4(1)\Big)$$

- Polynomial expression is valid for small m_D. For a generic value of m_D, the expression is continuated to tetra-logarithm.

Asymptotic entropy: II

- Asymptotic entropy of N M5-branes

$$\boxed{\frac{F}{\epsilon_1\epsilon_2} \simeq \frac{1}{N\pi i\epsilon_1\epsilon_2\tau}\Big(\frac{1}{2}\mathrm{Li}_4(e^{Nm}) + \frac{1}{2}\mathrm{Li}_4(e^{-Nm}) - \mathrm{Li}_4(1)\Big), \quad \tau = i\cdot 0^+}$$

- Imaginary m: Periodicity $m \sim m + i\frac{2\pi}{N}$

$$m = i\frac{2\pi}{N},\ m^D = \frac{2\pi i\tau^D}{N} \quad \to \quad e^{Nm^D} = q^D : \ \text{Phase transition point}$$

- Real m: N^3 scaling

$$\frac{F}{\epsilon_1\epsilon_2} \simeq \frac{i\pi^3}{3\epsilon_1\epsilon_2\tau}\left[N^3\Big(\frac{m}{2\pi}\Big)^4 - N\Big(\frac{m}{2\pi}\Big)^2\right]$$